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ON BIFURCATION AND STABILITY OF PERMANENT ROTATIONS OF A HEAVY SOLID BODY WHOSE CENTER OF MASS IS CLOSE TO THE PRINCIPAL PLANE OF ITS ELLIPSOID OF INERTIA*

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(This is the second part of the author's report on "Stability of steady motions of mechanical systems containing absolutely rigid, elastic, and fluid components" at the 5-th All-Union Congress on Theoretical and Applied Mechanics, Alma-Ata, 27 May 1981).

All possible qualitatively different types of bifurcation diagrams for bodies whose center of mass is close to the principal planes of their triaxial ellipsoids of inertia are classified, and the most interesting of such diagrams are presented.

Stability of permanent rotations of a heavy solid body was investigated in /1/ using Chetaev's method for constructing Liapunov functions /2/, and some of the domains corresponding to stable and unstable rotations were indicated on the Staude cone. The widest sufficient conditions of permanent rotation stability were derived in /3/ on the basis of the Routh—Liapunov theorem /4/. The bifurcation and stability of permanent rotations were investigated in /5—7/ in the case of a body whose center of mass lies on the principal axis of its triaxial ellipsoid of inertia and, also, in the case when the ellipsoid of inertia is an ellipsoid of revolution.

1. The equations of motion of a heavy solid body about a fixed point O admit the following energy and area integrals:

$$U=\frac{1}{2}\sum_{i=1}^{3}(J_i\omega_i^2+2e_i\gamma_i)=\text{const},\quad U_1=\sum_{i=1}^{3}J_i\omega_i\gamma_i=k=\text{const}$$

where ω_i are projections of the body angular velocity on the principal axes x_i of its ellipsoid on inertia relative to point O, J_i $(J_1<J_2<J_3)$ are the principal moments of inertia of the body, γ_i are cosines of angles between the upward vertical and axes x_i with $U_2=\gamma_1^2+\gamma_2^2+\gamma_3^2=1$, and e_i are constants equal to the products of the body weight by the coordinates of its center of mass.

The values of ω_i,γ_i for which U has fixed values under conditions $U_1=k$, $U_2=1$, correspond to permanent rotations and are defined by formulas /5—7/

$$\omega_1=\omega\gamma_1,\quad \gamma_1=\frac{e_1}{\omega^2(J_1-\lambda)}\quad (1\,2\,3)\quad (\lambda\ \text{is a parameter}) \tag{1.1}$$

with the dependence of the angular velocity ω of permanent rotation on parameter k determined by formulas

$$\omega=\pm\Big(\sum_{(123)}\frac{e_1^2}{(J_1-\lambda)^2}\Big)^{1/4},\quad k=\pm\Big(\sum_{(123)}\frac{e_1^2}{(J_1-\lambda)^2}\Big)^{-3/4}\sum_{(123)}\frac{J_1e_1^2}{(J_1-\lambda)^2} \tag{1.2}$$

The sufficient conditions of stability of motion (1.1) relative to ω_i,γ_i reduce to the inequalities /3,5—7/

$$\begin{gathered}L>0,\quad \Delta=\omega^2(4L+JS)>0\\ L=\sum_{(123)}(\lambda-J_1)(J_2-J_3)^2\gamma_2^2\gamma_3^2,\quad S=\sum_{(123)}(\lambda-J_2)(\lambda-J_3)\gamma_1^2,\quad J=\sum_{(123)}J_1\gamma_1^2\end{gathered} \tag{1.3}$$

The equation $L(\lambda)=0$ has a single real root /7/ $\lambda=\lambda^\circ$, and $J_1\leqslant\lambda^\circ\leqslant J_2$, when $(J_2-J_1)e_3^2>(J_3-J_2)e_1^2$, while $J_2\leqslant\lambda^\circ\leqslant J_3$ when $(J_2-J_1)e_3^2<(J_3-J_2)e_1^2$, and $\operatorname{sgn}L(\lambda)=\operatorname{sgn}(\lambda-\lambda^\circ)$.

When conditions (1.3) are satisfied, motion (1.1) is stable, and its instability degree $\chi=0$. Condition $\Delta>0$ is also the necessary stability condition /7/; motions (1.1) for which $\Delta<0$, are unstable, and for them $\chi=1$. If conditions (1.3) are not satisfied and $\Delta>0$, then $\chi=2$.

Motions (1.1) can be geometrically represented by points of curve $k=k(\lambda)$ which is determined by the second of Eqs.(1.2).

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The formula (1.3) for Δ can be represented in the form /5 – 7/

$$\Delta = 2\omega\varphi(\lambda)dk/d\lambda, \quad \varphi(\lambda) = (J_1 - \lambda)(J_2 - \lambda)(J_3 - \lambda) \tag{1.4}$$

which enables us to reduce the analysis of the sign of Δ to the determination of the form of curve $k = k(\lambda)$. Equation $\Delta(\lambda) = 0$ is used for determining bifurcation points /5 – 7/ and is equivalent to the equation $dk/d\lambda = 0$; at these points the tangent to curve $k = k(\lambda)$ is parallel to the λ axis.

Since conditions (1.3) are not affected by the substitution of $-\omega$ for ω, it is possible to restrict the investigation of distribution of steady and unsteady motions (1.1) on curve $k = k(\lambda)$ to the investigation of its branches for which $k > 0$ $(\omega > 0)$.

The form of curve $k = k(\lambda)$ and the degree of instability distribution $\chi = 0, 1, 2$ on it for $4J_1 \geqslant 3J_3$, $e_1e_2e_3 \neq 0$ appear in /7/.

Equation $dk/d\lambda = 0$ is equivalent to the equation

$$P_{10}(\lambda) = \sum_{(123)} \{J_1(J_2 - \lambda)^5(J_3 - \lambda)^5 e_1^4 + \varphi^2(\lambda)(J_1 - \lambda)^3[(4J_2 - 3J_3)(J_3 - \lambda) + (4J_3 - 3J_2)(J_2 - \lambda)]e_2^2e_3^2\} = 0 \tag{1.5}$$

The problem of separating real roots of Eq.(1.5) is generally difficult, since known methods of analysis /8/, for instance, the device of Sturm series or the Routh algorithm require the determination of the number of sign changes of fairly unwieldy expressions in (1.5), which depend on many parameters. If the body center of mass lies in the principal plane of its ellipsoid of inertia, Eq.(1.5) is considerably simplified and the problem of separating real roots can be completely solved. When that problem is solved, it is possible to draw definite conclusions on the bifurcations of motions (1.1), when the body center of mass is close to the principal plane of the ellipsoid of inertia.

2. Let $e_1e_2 \neq 0$, $e_3 = 0$, then (1.1) and (1.2) assume the form

$$\omega_i = \omega\gamma_i, \quad \omega^2\gamma_j = e_j(J_j - \lambda)^{-1}, \quad \omega^2\gamma_3 = \varkappa\delta(\lambda - J_3), \quad (i = 1, 2, 3; j = 1, 2) \tag{2.1}$$

$$\omega^4 = \varkappa^2\delta(\lambda - J_3) \cdot \sum_j \frac{e_j^2}{(J_j - \lambda)^2}$$

$$k = k(\lambda, \varkappa) = \left[J_3\varkappa^2\delta(\lambda - J_3) + \sum_j \frac{J_je_j^2}{(J_j - \lambda)^2}\right] \times \left[\varkappa^2\delta(\lambda - J_3) + \sum_j \frac{e_j^2}{(J_j - \lambda)^2}\right]^{-1/4}$$

where $\delta(x) = 0$ when $x \neq 0$, $\delta(0) = 1$, and $\varkappa$ is a variable real parameter.

Motion (2.1) can be geometrically represented /5/ in the space of parameters $k, \lambda, \varkappa$ by points of manifold $k = k(\lambda, \varkappa)$ consisting of a cylindrical surface $k = k(\lambda, 0)$ and curve $k = k(J_3, \varkappa)$, $\lambda = J_3$ which have one common point $\varkappa = 0, \lambda = J_3, k = k(J_3, 0)$. Since one and only one motion (2.1) corresponds to each generatrix of the cylindrical surface, we juxtapose to these generatrices points of their intersection with plane $\varkappa = 0$. To represent motions (2.1) we introduce in the analysis curve Γ whose branches lie in planes $\varkappa = 0$ and $\lambda = J_3$ and are defined by equations $k = k(\lambda, 0), \varkappa = 0$ and $k = k(J_3, \varkappa), \lambda = J_3$, respectively.

To investigate curve $k = k(\lambda, 0), \varkappa = 0$ we use the equation $\partial k/\partial\lambda = 0$ which is equivalent to equation

$$\begin{aligned} &P_{10}(\lambda) = (J_3 - \lambda)^5 P_5(\lambda) \\ &P_5(\lambda) = J_1(J_2 - \lambda)^5 e_1^4 + J_2(J_1 - \lambda)^5 e_2^4 + [(4J_1 - 3J_2)(J_2 - \lambda) + (4J_2 - 3J_1)(J_1 - \lambda)](J_1 - \lambda)^2(J_2 - \lambda)^2 e_1^2e_2^2 \end{aligned} \tag{2.2}$$

whose root $\lambda = J_3$ is of multiplicity 5. To investigate the roots of polynomial $P_5(\lambda)$ we substitute the new variable z for λ and introduce the notation

$$z = \frac{J_1 - \lambda}{J_2 - \lambda}, \quad a = \frac{J_1}{J_2}, \quad b = \frac{J_3}{J_2}, \quad e = \frac{e_1}{e_2} \tag{2.3}$$

then $P_5(\lambda)$ reduces to the form

$$P_5(\lambda) \equiv P(\lambda) = z^5 - e^2z^2\Phi(z) + ae^4 = 0, \quad \Phi(z) = (3a - 4)z + 3 - 4a \tag{2.4}$$

3. Consider the problem of separating real roots of Eq.(2.4). By the Descartes theorem Eq.(2.4) has one real root $z_1 < 0$ when $a \geqslant 3/4$; if $a < 3/4$, (2.4) has one negative root z_1 and two or none positive roots z_2, z_3 $(z_3 \geqslant z_2)$. Hence we assume in what follows that $a < 3/4$.

We introduce the two-valued function $e^2 = F(z)$ defined by Eq.(2.4). For branches $e^2 = F^{(\pm)}(z)$ of that function we have

$$F^{(\pm)}(z) = \frac{z^2}{2a}\left[\Phi(z) \pm \sqrt{\Phi^2(z) - 4az}\right]$$

Function $e^2 = F(z)$ has real values only when $\Phi^2(z) \geqslant 4az$. If it is not to assume negative values it is necessary that the following condition is satisfied:

$$z \leqslant z_*, \quad z_* = (3 - 4a)(4 - 3a)^{-1} \quad (a < {}^3/_4)$$

We denote the roots of equation $\Phi^2(z) - 4az = 0$ by z_{1*} and z_{2*} $(z_{1*} < z_{2*})$. We have an array of inequalities $0 < z_{1*} < z_* < z_{2*}$. This implies that function $e^2 = F(z)$ assumes real values and is nonnegative only for $0 \leqslant z \leqslant z_{1*}$, while for $z < 0$ the branch $e^2 = F^{(+)}(z)$ assumes only positive and branch $e^2 = F^{(-)}(z)$ only negative values. The form of curve $e^2 = F(z)$ is shown in Fig.1, where only branches that do not assume negative values are plotted.

From this we draw the conclusion that for each value of $a\ (a < {}^3/_4)$ Eq.(2.4) has three real roots, viz. z_1, z_2, z_3 $(z_1 < 0 < z_2 \leqslant z_3)$, if $0 < e^2 \leqslant F(z_{1*})$, and the equality $z_2 = z_3$ holds only when $e^2 = F(z_{1*})$; if $e^2 > F(z_{1*})$, then (2.4) has one real root $z_1 < 0$.

Let us take some number $z_0 > 0$ and assume that $z_{1*} < z_0$, then (Fig.1) Eq.(2.4) with $0 < e^2 \leqslant F(z_{1*})$ has three real roots: z_1, z_2, z_3 $(z_1 < 0 < z_2 < z_3 \leqslant z_{1*} < z_0)$, and the equality $z_3 = z_{1*}$ holds only for $e^2 = F(z_{1*})$, and when $z_2 = z_3 = z_{1*}$; if however $e^2 > F(z_{1*})$, then (2.4) has one real root $z_1 < 0$.

If now $z_{1*} > z_0$, then Eq.(2.4) for $0 < e^2 \leqslant F(z_{1*})$ has three real roots: z_1, z_2, z_3 $(z_1 < 0 < z_2 \leqslant z_3)$, and (Fig.1)

$$\begin{array}{ll}
z_2 < z_3 < z_0 < z_{1*}, & 0 < e^2 < F^{(-)}(z_0) \\
z_2 < z_3 = z_0 < z_{1*}, & e^2 = F^{(-)}(z_0) \\
z_2 < z_0 < z_3 < z_{1*}, & F^{(-)}(z_0) < e^2 < F^{(+)}(z_0) \\
z_2 = z_0 < z_3 < z_{1*}, & e^2 = F^{(+)}(z_0) \\
z_0 < z_2 < z_3 < z_{1*}, & F^{(+)}(z_0) < e^2 < F(z_{1*}) \\
z_0 < z_2 = z_3 = z_{1*}, & e^2 = F(z_{1*})
\end{array}$$

if however $e^2 > F(z_{1*})$, then (2.4) has one real root $z_1 < 0$.

4. Consider the question of the number and disposition of real roots of the polynomial $P_5(\lambda)$. We denote by λ_i $(i = 1, 2, 3)$ the values of λ which by virtue of (2.3) correspond to values z_i and set $z_0 = (J_1 - J_3)(J_2 - J_3)^{-1} = (a - b)(1 - b)^{-1} > 1$. On the basis of the analysis in Sect.3 we conclude then that for $a < 3/4$

$$\begin{array}{l}
1^\circ.\ \lambda_3 < \lambda_2 < J_1 < \lambda_1 < J_2, \quad 0 < e^2 < F(z_{1*}) \\
2^\circ.\ \lambda_3 = \lambda_2 < J_1 < \lambda_1 < J_2, \quad e^2 = F(z_{1*}) \\
3^\circ.\ J_1 < \lambda_1 < J_2, \quad e^2 > F(z_{1*})
\end{array} \qquad (4.1)$$

For $a \geqslant {}^3/_4$ the polynomial $P_5(\lambda)$ has one real root λ_1, $J_1 < \lambda_1 < J_2$.

Investigation of stability of motions (2.1) in the cases (4.1) does not present difficulties and is carried out as in /5–7/.

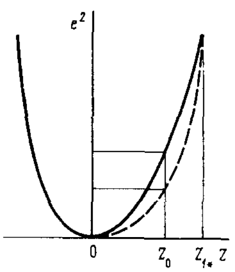

Fig.1

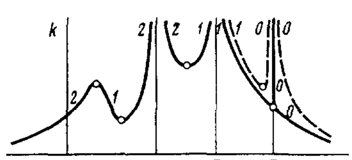

Fig.2

Projections of curve Γ on the plane $\varkappa = 0$ are shown in Fig.2 (solid lines) for case 1° in (4.1); the rectilinear branch for $\lambda = J_3$ represents there the projection on plane $\varkappa = 0$ of the branch $k = k(J_3, \varkappa)$, $\lambda = J_3$ of curve Γ, and the numerals 0,1,2 indicate the instability degree of respective motions. In case 2° the respective curve differs from that in Fig.2 only in that it has an inflection point when $\lambda = \lambda_2$ at which the instability degree remains unchanged, and in case 3° that curve increases strictly monotonically when $\lambda < J_1$. The form of branch $k = k(J_3, \varkappa)$, $\lambda = J_3$ for all cases of (4.1) is the same as in /7/.

5. Let now the center of mass lie close to the plane $x_3 = 0$. Then e_3 is a small quantity. Introducing in addition to (2.3) the notation

$$x = (J_3 - \lambda)(J_2 - \lambda)^{-1}, \quad z = z(x) = [a - b + (1 - a)x](1 - b)^{-1}, \quad \varepsilon = e_3 e_2^{-1} \tag{5.1}$$

we represent Eq.(1.5) in the form

$$G(x, \varepsilon) = x^5 P_5(z) - \varepsilon^2 x^2 z^2 \{[(3b - 4)z^3 + (3b - 4a)e^2]x + zR(z)\} + \varepsilon^4 b z^5 = 0 \tag{5.2}$$

$$R(z) = (3 - 4b)z^2 + (3a - 4b)e^2 \tag{5.3}$$

Equation (5.2) defines x as an implicit function of parameter ε. For $\varepsilon = 0$ it has the root $x = 0$ of multiplicity five. For finding real branches of $x = x(\varepsilon)$ we use the Newton diagram in /9/. We shall seek $x = x(\varepsilon)$ of the form

$$x = c\varepsilon^n + c'\varepsilon^{n+1} + \ldots \tag{5.4}$$

Substituting (5.4) into (5.2) and taking into account (5.1), we equate the totality of lower order terms to zero and obtain the relation

$$\varepsilon^{5n} c^5 P_5(z_0) - \varepsilon^{2n+2} c^2 z_0^3 R(z_0) + \varepsilon^4 b z_0^5 = 0, \quad z_0 = \frac{b - a}{b - 1} (> 1)$$

from which we have

$$\text{a) } n = \frac{2}{3}, \quad c^3 = \frac{z_0^3 R(z_0)}{P_5(z_0)}; \quad \text{b) } n = 1, \quad c^2 = \frac{b z_0^2}{R(z_0)} \tag{5.5}$$

Consider case 1° of (4.1) in which $R(z_0) < 0$, $P_5(z_0) > 0$. On the basis of (5.5) we conclude that Eq.(5.2) in the neighborhood of $x = 0$, $\varepsilon = 0$ defines one real branch $x = x^*(\varepsilon) < 0$, $x^*(\varepsilon) \to 0$, as $\varepsilon \to 0$. The value $\lambda = \lambda^*(\varepsilon)$, $J_2 < \lambda^*(\varepsilon) < J_3$, $\lambda^*(\varepsilon) \to J_3$ corresponds by virtue of (5.1) to $x = x^*(\varepsilon)$, as $\varepsilon \to 0$. Curve $k = k(\lambda)$ is shown in Fig.2 (the dash lines relate $\lambda > J_2$) for small $e_3 \neq 0$ for case 1° of (4.1).

6. If now $e_3 e_1 \neq 0$, $e_2 = 0$, then relations (1.1) and (1.2) assume the form

$$\omega_i = \omega\gamma_i, \quad \omega^2\gamma_j = e_j (J_j - \lambda)^{-1}, \quad \omega^2\gamma_2 = \varkappa\delta(\lambda - J_2) \quad (i = 1, 2, 3; \ j = 1, 3) \tag{6.1}$$

$$\omega^4 = \varkappa^2\delta(\lambda - J_2) + \sum_j \frac{e_j^2}{(J_j - \lambda)^2}$$

$$k = k(\lambda, \varkappa) = \left[J_2\varkappa^2\delta(\lambda - J_2) + \sum_j \frac{J_j e_j^2}{(J_j - \lambda)^3}\right] \times \left[\varkappa^2\delta(\lambda - J_2) + \sum_j \frac{e_j^2}{(J_j - \lambda)^2}\right]^{-3/4}$$

For motions (6.1) the branches of curve Γ are defined by the equations $k = k(\lambda, 0)$, $\varkappa = 0$ and $k = k(J_2, \varkappa)$, $\lambda = J_2$.

Let us investigate the form of curve $k = k(\lambda, 0)$, $\varkappa = 0$. The equation $\partial k/\partial\lambda = 0$ is equivalent to the equation of form (2.2) in which we have now

$$P_5(\lambda) = J_3 (J_1 - \lambda)^5 e_3^4 + J_1 (J_3 - \lambda)^5 e_1^4 + [(4J_3 - 3J_1)(J_1 - \lambda) + (4J_1 - 3J_3)(J_3 - \lambda)]^2 (J_3 - \lambda)^2 (J_1 - \lambda)^2 e_3^2 e_1^2 \tag{6.2}$$

Introducing the notation

$$z = \frac{J_1 - \lambda}{J_3 - \lambda}, \quad x = \frac{J_2 - \lambda}{J_3 - \lambda}, \quad a = \frac{J_1}{J_3}, \quad b = \frac{J_2}{J_3}, \quad e = \frac{e_1}{e_3}, \quad \varepsilon = \frac{e_2}{e_3} \ (a < b < 1) \tag{6.3}$$

we reduce equation $P_5(\lambda) = 0$ to the form (2.4). Setting as in the capacity of z_0 the value $z_0 = (J_1 - J_2)(J_3 - J_2)^{-1} = (a - b)(1 - b)^{-1} < 0$ and using the results of Sect.3, we conclude that the number and disposition of real roots of polynomial (6.2) with $a < 3/4$ are:

$$
\begin{array}{lll}
1^\circ. & \lambda_3 < \lambda_2 < J_1 < \lambda_1 < J_2, & 0 < e^2 < F^{(+)}(z_0) < F(z_{1*}) \\
2^\circ. & \lambda_3 < \lambda_2 < J_1 < \lambda_1 = J_2, & e^2 = F^{(+)}(z_0) < F(z_{1*}) \\
3^\circ. & \lambda_3 < \lambda_2 < J_1 < J_2 < \lambda_1 < J_3, & F^{(+)}(z_0) < e^2 < F(z_{1*}) \\
4^\circ. & \lambda_3 = \lambda_2 < J_1 < J_2 < \lambda_1 < J_3, & F^{(+)}(z_0) < e^2 = F(z_{1*}) \\
5^\circ. & J_2 < \lambda_1 < J_3, & F^{(+)}(z_0) < F(z_{1*}) < e^2 \\
6^\circ. & \lambda_3 < \lambda_2 < J_1 < \lambda_1 < J_2, & 0 < e^2 < F^{(+)}(z_0) = F(z_{1*}) \\
7^\circ. & \lambda_3 = \lambda_2 < J_1 < \lambda_1 = J_2, & e^2 = F^{(+)}(z_0) = F(z_{1*}) \\
8^\circ. & J_2 < \lambda_1 < J_3, & e^2 > F^{(+)}(z_0) = F(z_{1*}) \\
9^\circ. & \lambda_3 < \lambda_2 < J_1 < \lambda_1 < J_2, & 0 < e^2 < F(z_{1*}) < F^{(+)}(z_0) \\
10^\circ. & \lambda_3 = \lambda_2 < J_1 < \lambda_1 < J_2, & e^2 = F(z_{1*}) < F^{(+)}(z_0) \\
11^\circ. & J_1 < \lambda_1 < J_2, & F(z_{1*}) < e^2 < F^{(+)}(z_0) \\
12^\circ. & \lambda_1 = J_2, & F(z_{1*}) < e^2 = F^{(+)}(z_0)
\end{array} \tag{6.4}
$$

$13°.\ J_2<\lambda_1<J_3$, for $a\geqslant 3/4$	$e^2>F^{(+)}(z_0)>F(z_{1*})$
$14°.\ J_1<\lambda_1<J_2$,	$0<e^2<F^{(+)}(z_0)$
$15°.\ \lambda_1=J_2$,	$e^2=F^{(+)}(z_0)$
$16°.\ J_2<\lambda_1<J_3$,	$e^2>F^{(+)}(z_0)$

Solid lines in Fig.3 show the form of projections of curve Γ on the plane $\varkappa=0$ for cases 1°, 6°, 9° of (6.4). The respective curve for case 10° has an inflection point when $\lambda=\lambda_2$. Curves for cases 5°, 8°, $11^\circ-16^\circ$ are given in /7/. For all cases of (6.4) the form of curve $k=k(J_2,\varkappa)$, $\lambda=J_2$ is the same as in /7/.

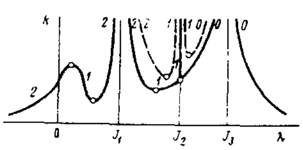

Fig.3

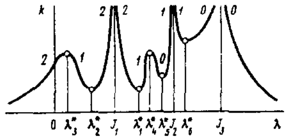

Fig.4

Let now the center of mass lie close to the plane $x_2=0$. Then e_2 is a small quantity, and Eq.(1.5) with notation (6.3) can be represented in the form (5.2), where now $z=[a-b+(1-a)\,x](1-b)^{-1}$.

We represent formula (5.3) for $z=z_0$ in the form

$$R(z_0)=(4b-3a)(K(z_0)-e^2),\quad K(z_0)=z_0^2(3-4b)(4b-3a)^{-1} \tag{6.5}$$

The following inequality holds:

$$F^{(+)}(z_0)>K(z_0)\quad (0<a<3/4) \tag{6.6}$$

To establish the form of curve $k=k(\lambda)$ we use the results of Sects.3 and 4, and formulas (5.3)—(5.5), (6.5) and (6.6).

For small ε we have in cases 1° and 6° of (6.4) $R(z_0)>0$, $P_5(z_0)<0$, when $0<e^2<K(z_0)$ (Fig. 4), and $R(z_0)<0$, $P_5(z_0)<0$ when $K(z_0)<e^2<F^{(+)}(z_0)$ (dash lines in Fig.3 relate to $J_1<\lambda<J_3$). For case 9° we have $R(z_0)>0$, $P_5(z_0)<0$ when $0<e^2<K(z_0)<F(z_{1*})$ (Fig.4), and $R(z_0)<0$, $P_5(z_0)<0$ when $K(z_0)<e^2<F(z_{1*})$ (dash lines in Fig.3 relate to $J_1<\lambda<J_3$).

Let us investigate the stability of motions (1.1), when $\varepsilon=e_2/e_3$ is a small parameter and conditions 1° of (6.4) and also the supplementary condition $0<e^2<K(z_0)$, under which curve $k=k(\lambda)$ is of the form shown in Fig.4, are satisfied.

We denote by $\lambda_\nu^*(\varepsilon)$ ($\nu=1,\ldots,6$) the real roots of Eq.(1.5) and, using the results of Sect. 3 and formulas (5.3)—(5.5) we obtain for them the following expressions:

$$\lambda_i^*(\varepsilon)=\lambda_i+o(\varepsilon)\quad (i=1,\ 2,\ 3) \tag{6.7}$$

$$\lambda_4^*(\varepsilon)=J_2+(J_3-J_2)\,|z_0|\left[\frac{R(z_0)}{P_5(-|z_0|)}\right]^{1/3}\varepsilon^{2/3}+o(\varepsilon^{2/3})$$

$$\lambda_{5,6}^*(\varepsilon)=J_2\mp(J_3-J_2)\,|z_0|\left[\frac{b}{R(z_0)}\right]^{1/2}\varepsilon+o(\varepsilon)$$

$$\lambda_3^*(\varepsilon)<\lambda_2^*(\varepsilon)<J_1<\lambda_1^*(\varepsilon)<\lambda_4^*(\varepsilon)<\lambda_5^*(\varepsilon)<J_2<\lambda_6^*(\varepsilon)<J_3$$

We shall now consider the equation $L(\lambda)=0$, which with allowance for (6.3) we represent in the form

$$Q(x,\ \varepsilon)=(1-a)^2e^3x^3+\varepsilon^2\left[(1-b)^2z^3+(a-b)^2e^2\right]=0 \tag{6.8}$$

which for $\varepsilon=0$ has the triple root $x=0$. In the case of small ε we seek a real root of Eq.(6.8) of the form

$$x=x^\circ(\varepsilon)=\alpha\varepsilon^n+o(\varepsilon^n) \tag{6.9}$$

Substituting (6.9) into (6.8) we obtain

$$n=\frac{2}{3},\quad \alpha^3=\frac{(|z_0|-e^2)(1-b)^2z_0^2}{(1-a)^2e^3} \tag{6.10}$$

From (6.9), (6.10), and (6.3) we obtain for the real root $\lambda=\lambda^\circ(\varepsilon)$ of the equation $L(\lambda)=0$ the expression

$$\lambda^\circ(\varepsilon)=J_2-(J_3-J_2)(|z_0|-e^2)^{1/3}\left[\frac{(1-b)\,|z_0|}{(1-a)\,e}\right]^{2/3}\varepsilon^{2/3}+o(\varepsilon^{2/3}) \tag{6.11}$$

We pass to the analysis of stability conditions (1.3). Using formula (1.4) and taking into account the form of curve $k=k(\lambda)$ (Fig.4), we conclude that $\Delta>0$, if $\lambda<\lambda_3^*$, $\lambda_2^*<\lambda<J_1$, $\lambda_4^*<\lambda<\lambda_5^*$ or $\lambda>\lambda_6^*$. Moreover, it follows from (6.11) that $\lambda^\circ(\varepsilon)<J_2$, hence $L>0$, if $\lambda>J_2$, and $L<0$, if $\lambda<\lambda^\circ(\varepsilon)$. Hence motions (1.1) are stable $\chi=0$, if $\lambda>\lambda_6^*$, and unstable $\chi=1$ if $\lambda_5^*<\lambda<\lambda_6^*$, $\lambda_1^*<\lambda<\lambda_5^*$ or $\lambda_3^*<\lambda<\lambda_2^*$. If however $\lambda<\lambda_3^*$, $\lambda_2^*<\lambda<J_1$ or $J_1<\lambda<\lambda_1^*$ then $\chi=2$.

Remains to investigate the stability of motions (1.1) for λ from the interval $\lambda_4^* < \lambda < \lambda_5^*$. For that it is sufficient to determine the sign of quantity $L(\lambda)$.

From (6.7) and (6.11) we can see that $\lambda_4^*(\varepsilon)$ and $\lambda^\circ(\varepsilon)$ have the same order of smallness, while $\lambda_5^*(\varepsilon)$ has a large order of smallness. Therefore $L > 0$ for the values of λ close to λ_5^* $(\lambda < \lambda_5^*)$ and motions (1.1) are stable $\chi = 0$. Since in the case of continuous variation of λ from its value at which stability conditions (1.3) are satisfied, first condition $\Delta > 0$ is violated /2/, hence we conclude that $\lambda^\circ(\varepsilon) \leqslant \lambda_4^*(\varepsilon)$ and for $\lambda_4^* < \lambda < \lambda_5^*$ motions (1.1) are stable $\chi = 0$.

We would point out that one of the cases whose bifurcation diagram appears in Fig.4 was investigated in /10/.

7. Finally, let $e_2 e_3 \neq 0$, $e_1 = 0$, then

$$\omega_i = \omega\gamma_i, \quad \omega^2\gamma_j = e_j (J_j - \lambda)^{-1}, \quad \omega^2\gamma_1 = \varkappa\delta(\lambda - J_1) \quad (i = 1, 2, 3; \; j = 2, 3) \tag{7.1}$$

$$\omega^4 = \varkappa^2\delta(\lambda - J_1) + \sum_j \frac{e_j^2}{(J_j - \lambda)^2}$$

$$k = k(\lambda, \varkappa) = \Big[J_1\varkappa^2\delta(\lambda - J_1) + \sum \frac{J_j e_j^2}{(J_j - \lambda)^2}\Big] \times \Big[\varkappa^2\delta(\lambda - J_1) + \sum_j \frac{e_j^2}{(J_j - \lambda)^2}\Big]^{-3/4}$$

In the case of motions (7.1) the branches of curve Γ are defined by the equations $k = k(\lambda, 0)$, $\varkappa = 0$ and $k = k(J_1, \varkappa), \lambda = J_1$. Let us investigate the form of the first of these curves. The equation $\partial k / \partial \lambda = 0$ is equivalent to an equation of the form (2.2) in which now

$$P_5(\lambda) = J_2 (J_3 - \lambda)^5 e_2^4 + J_3 (J_2 - \lambda)^5 e_3^4 + [(4J_2 - 3J_3)(J_3 - \lambda) + (4J_3 - 3J_2)(J_2 - \lambda)](J_2 - \lambda)^2 (J_3 - \lambda)^2 e_2^2 e_3^2 \tag{7.2}$$

Introducing the notation

$$z = \frac{J_2 - \lambda}{J_3 - \lambda}, \quad x = \frac{J_1 - \lambda}{J_3 - \lambda}, \quad a = \frac{J_2}{J_3}, \quad b = \frac{J_1}{J_3}, \quad e = \frac{e_2}{e_3}, \quad \varepsilon = \frac{e_1}{e_3} \quad (b < a < 1) \tag{7.3}$$

we reduce equation $P_5(\lambda) = 0$ to the form (2.4).

We set $z_0 = (J_2 - J_1)(J_3 - J_1)^{-1} = (a - b)(1 - b)^{-1} > 0$. For $a < 3/4$ the inequality $z_{1*}(a) < a$ is valid, hence the equation $z_{1*}(a) = z_0$ has a single real root $b = b_*(a)$, $0 < b_*(a) < a$, and $z_{1*}(a) < z_0$ when $b < b_*$, and $z_{1*}(a) > z_0$, when $b_* < b < a < 3/4$. From this on the basis of results in Sect.1 we arrive to the following conclusions about the number and disposition of real roots of polynomial (7.2):

for $a < 3/4$ (7.4)

1° $J_1 < \lambda_3 < \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} < z_0$, $0 < e^2 < F(z_{1*})$

2° $J_1 < \lambda_3 = \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} < z_0$, $e^2 = F(z_{1*})$

3°. $J_2 < \lambda_1 < J_3$, $z_{1*} < z_0$, $e^2 > F(z_{1*})$

4°. $J_1 < \lambda_3 < \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} = z_0$, $0 < e^2 < F(z_{1*})$

5°. $J_1 = \lambda_3 = \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} = z_0$, $e^2 = F(z_{1*})$

6° $J_2 < \lambda_1 < J_3$, $z_{1*} = z_0$, $e^2 > F(z_{1*})$

7°. $J_1 < \lambda_3 < \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $0 < e^2 < F^{(-)}(z_0)$

8°. $J_1 = \lambda_3 < \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $e^2 = F^{(-)}(z_0)$

9°. $\lambda_3 < J_1 < \lambda_2 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $F^{(-)}(z_0) < e^2 < F^{(+)}(z_0)$

10°. $\lambda_3 < \lambda_2 = J_1 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $F^{(+)}(z_0) = e^2$

11°. $\lambda_3 < \lambda_2 < J_1 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $F^{(+)}(z_0) < e^2 < F(z_{1*})$

12°. $\lambda_3 = \lambda_2 < J_1 < J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $e^2 = F(z_{1*})$

13° $J_2 < \lambda_1 < J_3$, $z_{1*} > z_0$, $e^2 > F(z_{1*})$

and for $a \geqslant 3/4$ 14° $J_2 < \lambda_1 < \lambda_3$

For motions (7.1) to which correspond points of curve $k = k(J_1, \varkappa), \lambda = J_1$ we have $L < 0$ if $\varkappa \neq 0$, and Δ can be represented in the form

$$\Delta = 2\varkappa\omega (J_2 - J_1)(J_3 - J_1)\frac{\partial k}{\partial \varkappa}, \quad \frac{\partial k}{\partial \varkappa} = \frac{\varkappa}{2\omega^7}\left(J_1\varkappa^2 - \frac{R(z_0)\, e_3^2}{J_2 - J_1}\right) \tag{7.5}$$

Formula (5.3) for $R(z_0)$ is now of the form

$$R(z_0) = (4b - 3a)(K(z_0) - e^2), \quad K(z_0) = z_0^2 (3 - 4b)(4b - 3a)^{-1} \tag{7.6}$$

and the following inequality holds:

$$F^{(+)}(z_0) < K(z_0) \quad (3/4a < b < a < 3/4), \quad F(z_{1*}) < K(z_0) (3/4a < b < b_* < a < 3/4) \tag{7.7}$$

Now, using (7.5)—(7.7), (1.3), and (1.4) we can establish the form of curve Γ and the distribution along it of the degree of instability of motions (7.1).

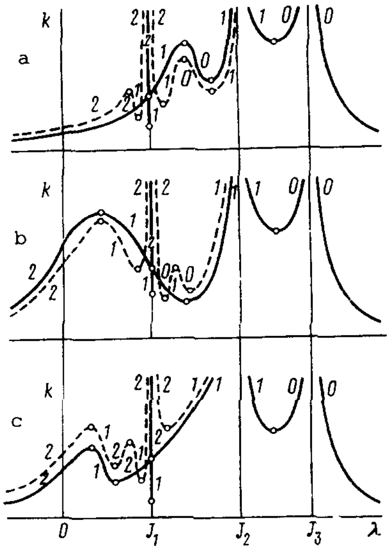

Fig.5

The form of curve $k = k(\lambda, 0)$, $\varkappa = 0$ is shown in Fig. 5,a (solid lines) for cases 1°, 4°, 7° of (7.4), and in Figs.5,b and c the solid lines relate, respectively, to 9°, and 11°, when $K(z_0) < F(z_{1*})$ and $e^2 < K(z_0)$, or when $K(z_0) > F(z_{1*})$.

The form of curve $k = k(J_1, \varkappa)$, $\lambda = J_1$ for all cases of (4.7) appeared in /7/.

Consider the case when the center of mass of the body lies close to the plane $x_1 = 0$, e_1 is then a small quantity, and Eq.(1.5) with notation (7.3) can be represented in the form (5.2), where now

$$z = [a - b + (1 - a)x](1 - b)^{-1}$$

Consider the equation $L(\lambda) = 0$ which with notation (7.3) can be represented in the form (6.8). For the real root $\lambda = \lambda^\circ(\varepsilon)$ of equation $L(\lambda) = 0$ we have the formula

$$\lambda^\circ(\varepsilon) = J_1 + (J_3 - J_1)\left[\frac{(1-b)^2(z_0 + e^2)\,z_0^2}{(1-a)^2 e^2}\right]^{1/3} \varepsilon^{2/3} + o(\varepsilon^{2/3}) \qquad (7.8)$$

Formulas (5.3)–(5.5), (7.8), (1.3), and (1.4) enable us to draw some conclusions about the form of curve $k = k(\lambda)$ and the distribution along it of the degree of instability of motions (1.1) for all cases of (7.4) for small values of $\varepsilon = e_1/e_3$.

The form of curve $k = k(\lambda)$ is shown in Fig.5,c (solid lines for $\lambda < J_2$) for case 11° of (7.4) when $K(z_0) < F(z_{1*})$ and $e^2 < K(z_0)$, or when $K(z_0) > F(z_{1*})$. The form of curve $k = k(\lambda)$ for cases 1°, 4°, 7° is similar to that shown in Fig.4, except that the width with respect to λ of secular stability $(\chi = 0)$, which lies between $\lambda = J_1$ and $\lambda = J_2$, does not approach zero as $\varepsilon \to 0$.

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